Solution Sheet 10:

Problem 5: First recall that $\delta_2 f(\sigma,\tau,g) = \sigma.f(\tau,g) - f(\sigma\tau,g) + f(\sigma,\tau g) - f(\sigma,\tau)$

Using $\delta_2 f = 0$ we write,

$$g(\sigma,\tau) + nf(\sigma,\tau) = g(\sigma,\tau) + \sum_{g\in G} f(\sigma,\tau)$$
$$= \sum_{g\in G}\left(f(\sigma,g) + \sigma f(\tau,g) - f(\sigma\tau,g) + f(\sigma,\tau)\right) = 0$$

Now notice that $g(\sigma,\tau) = \delta_1\left(\sum_{g\in G} f(\sigma,g)\right)(\tau)$ therefore $[nf] = 0$ in $H^2(G,M) \Rightarrow nH^2(G,M) = 0$.

Problem 6:

(i) Let us show that $\delta(f_m) = 0$.

$$\delta(f(\sigma^i,\sigma^j))(\sigma^k) = \sigma^i f(\sigma^j.\sigma^k) - f(\sigma^{i+j},\sigma^k) + f(\sigma^i,\sigma^{j+k}) - f(\sigma^i,\sigma^j).$$

First suppose that $i+j \geq n$, if $j+k \geq n$ then $f(\sigma^{i+j},\sigma^k) = 0 = f(\sigma^i,\sigma^{j+k})$ and $\sigma^i f(\sigma^j,\sigma^k) = \sigma^i.m = m = f(\sigma^i,\sigma^j) \Rightarrow \delta f(\sigma^i,\sigma^j,\sigma^k) = 0$.
If $j+k < n$ then $\sigma^i f(\sigma^j,\sigma^k) = 0 = f(\sigma^{i+j},\sigma^k)$ and $f(\sigma^i,\sigma^{j+k}) = m = f(\sigma^i,\sigma^j)$ $\Rightarrow \delta f(\sigma^i,\sigma^j,\sigma^k) = 0$. The same reasoning works for $i+j<n$, this shows that $\delta f_m = 0$.

(ii) $m \to f_m$ induces a morphism $M^G \xrightarrow{\phi} \ker(\delta_2)$ now let $m \in \mathrm{im}(T)$ i.e $m = \sum_{i=0}^{n-1} \sigma^i.v$ for some $v\in M$. Let us show that $f_m \in \mathrm{im}(\delta_1)$.

$$f_m(\sigma^a,\sigma^b) = \begin{cases} 0 \text{ if } a+b<n \\ \sum_{i=0}^{n-1}\sigma^i v \text{ if } a+b \geq n\end{cases}$$

let g be a 1-cochain recall that

$\delta_1 g(\sigma^a,\sigma^b) = \sigma^a g(\sigma^b) - g(\sigma^{a+b}) + g(\sigma^a)$. Defining $g(\sigma^a) = \sum_{i=0}^{a-1}\sigma^i.v$ we see that $\delta_1 g = f_m$. Therefore $\phi(\mathrm{im}(T)) \subseteq \mathrm{im}(\delta_1)$

We obtain a morphism $\bar{\phi}: M^G/_{\mathrm{im}T} \longrightarrow H^2(G,M)$.

Let us start by showing that $M^G \to H^2(G,M)$ is surjective. Let $h\in\ker(\delta_2)$ let us show that h is cohomologous to f_m where $m = \sum_{i=0}^{n-1} h(\sigma^i,\sigma)$.

Now as $h\in\ker(\delta_2)$,

$$\delta_2 h(\sigma^i,\sigma^{j-1},\sigma^k) = \sigma^i h(\sigma^{j-1},\sigma^k) - h(\sigma^{i+j-1},\sigma^k) + h(\sigma^i,\sigma^{j+k-1}) - h(\sigma^i,\sigma^{j-1}) = 0$$

$\Rightarrow h(\sigma^i,\sigma^{j+k-1}) = h(\sigma^i,\sigma^{j-1}) - \sigma^i h(\sigma^{j-1},\sigma^k) + h(\sigma^{i+j-1},\sigma^k)$ setting $k=1$ we get
$h(\sigma^i,\sigma^j) = h(\sigma^i,\sigma^{j-1}) - \sigma^i h(\sigma^{j-1},\sigma) + h(\sigma^{i+j-1},\sigma)$

Using this relation now on $h(\sigma^i,\sigma^{j-1})$ we get
$h(\sigma^i,\sigma^j) = h(\sigma^i,\sigma^{j-2}) - \sigma^i h(\sigma^{j-2},\sigma) + h(\sigma^{i+j-2},\sigma) + \sigma^i h(\sigma^{j-1},\sigma) + h(\sigma^{i+j-1},\sigma)$
repeating this we get

$$h(\sigma^i,\sigma^j) = h(\sigma^i,1) - \sigma^i\sum_{\ell=0}^{j-1} h(\sigma^\ell,\sigma) + \sum_{\ell=0}^{i+j-1} h(\sigma^\ell,\sigma) - \sum_{\ell=0}^{i-1} h(\sigma^\ell,\sigma)$$

Claim: For any $f\in\ker(\delta_2)$ we can find a cohomologous g such that

$g(\sigma,1)=g(\sigma,1)=g(1,1)=0.$

Proof: let $f\in\ker(\delta_2)$, define $h(\sigma)=f(\sigma,1)$ and $g=f-\delta_1 h$ then clearly f and g are cohomologous. Now

$$\delta_1 h(\sigma,\tau)=\sigma h(\tau)-h(\sigma\tau)+h(\sigma)=\sigma f(\tau,1)-f(\sigma\tau,1)+f(\sigma,1)$$
$$=f(\sigma,1)$$

where the last equality holds as $f\in\ker\delta_2$. This shows that $g(\sigma,1)=0$.
It is easy to show that for $g\in\ker\delta_2$ we have $g(1,1)=g(\sigma,1)=g(1,\sigma)$ □.

Thanks to the claim we get $h(\sigma^i,1)=0 \Rightarrow$
$$h(\sigma^i,\sigma^j)=\ -\sigma^i\textstyle\sum_{\ell=0}^{j-1}h(\sigma^\ell,\sigma)+\sum_{\ell=0}^{i+j-1}h(\sigma^\ell,\sigma)-\sum_{\ell=0}^{i-1}h(\sigma^\ell,\sigma).$$

Now define g a 1-cochain as $g(\sigma^i)=\sum_{\ell=0}^{i-1}f(\sigma^\ell,\sigma)$. Finally, it can be shown that $f_m-h(\sigma^i,\sigma^j)=\delta_1 g(\sigma^i,\sigma^j)$. This shows that
$$M^G/_{\operatorname{im}T}\to H^2(G,M)\ \text{ is surjective.}$$

It remains to show that it is injective. To this end suppose that f_{m_1} and f_{m_2} are cohomologous. Then

$$f_{m_1}(\sigma^i,\sigma^j)=f_{m_2}(\sigma^i,\sigma^j)+\delta_1 g(\sigma^i,\sigma^j)=f_{m_2}(\sigma^i,\sigma^j)+\sigma^i.g(\sigma^j)-g(\sigma^{i+j})+g(\sigma^i).$$

For $j=1$ we get $f_{m_1}(\sigma^i,\sigma)-f_{m_2}(\sigma^i,\sigma)+g(\sigma^{i+1})-g(\sigma^i)=\sigma^i g(\sigma)$

Therefore,

$$T(g(\sigma))=\textstyle\sum_{i=0}^{n-1}\sigma^i g(\sigma)=\sum_{i=0}^{n-1}f_{m_1}(\sigma^i,\sigma)-f_{m_2}(\sigma^i,\sigma)+g(\sigma^{i+1})-g(\sigma^i)=m_1-m_2.$$

this shows that $M^G/_{\operatorname{im}T}\to H^2(G,M)$ is injective and finishes the proof.

Problem 7:

(i) The goal is to show that $\delta(g_m)=0$. Recall that

$$\delta g_m(\sigma^i,\sigma^j)=\ \sigma^i g_m(\sigma^j)-g_m(\sigma^{i+j})+g_m(\sigma^i).\ (*)$$

First suppose that $i+j<n$ then

$$(*)=\sigma^i\left(\textstyle\sum_{k=0}^{j-1}\sigma^k.m\right)-\sum_{k=0}^{i+j-1}\sigma^k.m+\sum_{k=0}^{i-1}\sigma^k.m=0.$$

Now if $i+j\geq n$ then,

$$(*)=\sigma^i(m+\_\_+\sigma^{j-1}m)-(m+\_+\sigma^{i+j-n-1}m)+\textstyle\sum_{k=0}^{i-1}\sigma^k.m$$
$$=N(m)=0.$$

(ii) By the first point, $m\mapsto g_m$ defines a map
$$\ker(T)\to\ker(\delta_1)\twoheadrightarrow H^1(G,M).$$
Now let us show that if $m\in\operatorname{im}D$ then $[g_m]=[0]\in H^1(G,M)$.
Write $m=\sigma n-n$ then $g_m(\sigma^i)=m+\sigma m+\_\_+\sigma^{i-1}m=\sigma^{i-1}n-n\in\delta_0(n)(\sigma^i)$ and $g_m=\delta n\Rightarrow[g_m]=0\in H^1(G,M)$. Therefore the above map induces a map
$$\ker T/_{\operatorname{im}D}\xrightarrow{\psi}H^1(G,M).\ \text{Let us show that } \psi \text{ is injective.}$$
Suppose that $[g_{m'}]=[g_m]$ then $g_{m'}(\sigma^i)-g_m(\sigma^i)=\delta_0(n)(\sigma^i)$ for some n thus $g_{m'}-g_m(\sigma)=m'-m=\sigma n-n\in\operatorname{im}D$. This shows that ψ is injective.

It remains to show that ψ is surjective. Let $f \in \ker \delta_1$ define $m = f(\sigma)$ and let us show that $[f] = [f_m] \in H^1(G,M)$.
As $\delta_1 f = 0$ we have $\sigma^{i-1} f(\sigma) - f(\sigma^i) + f(\sigma^{i-1}) = 0$ then

$$\begin{aligned} f(\sigma^i) &= \sigma^{i-1} f(\sigma) + f(\sigma^{i-1}) \\ &= \sigma^{i-1} f(\sigma) + \sigma^{i-2} f(\sigma) + f(\sigma^{i-2}) \\ &\vdots \\ &= \sigma^i f(\sigma) + \sigma^{i-2} f(\sigma) + \_\_ + f(\sigma) + f(1) = f_m(\sigma^i) \end{aligned}$$

This finishes the proof that $H^1(G,M) \cong \ker(T)/\mathrm{im} D$.

Problem 8:

(i) Let us show that $\star$ is associative $\Leftrightarrow$ f is a 2-cocycle.
It is enough to show this at the level of the basis elements.
Let $g, h, k \in G$, then

$$\begin{aligned}(ae_g \star be_h) \star c.e_k &= a g(b). f(g,h). e_{gh} \star c.e_k \\ &= a.g(b). f(g,h). (gh)(c) f(gh,k) e_{ghk}\end{aligned}$$

on the other hand

$$\begin{aligned} ae_g \star (be_h \star ce_k) &= ae_g \star (b.h(c).f(h,k) e_{hk} \\ &= ag(b.h(c).f(h,k)).f(g,hk) e_{ghk}\end{aligned}$$

so we should show that

$$a.g(b).gh(c).gf(h,k). f(g,hk) = a(g.b). f(g,h).(gh.c) f(gh,k)$$

it is equiv. to showing that

$$gf(h,k). f(g,hk) = f(g,h). f(gh,k)$$

which is exactly the 2-cocycle condition.

(ii) We define a morphism of K-algebras $\phi: V_f \to M_n(K)$ by $\phi(e_\sigma) = \sigma$ and extending it K-linearly. Now to see that ϕ is an iso. note that $\dim_L V_f = |G|$ and $\dim_K L = |G|$ therefore $\dim_K V_f = |G|^2 = \dim_K M_n K$ and it suffices to show that ϕ is injective.
Suppose that $\phi(\sum a_\sigma e_\sigma) = \sum a_\sigma \sigma = 0$, by Dedekind's independence lemma this implies that $a_\sigma = 0$ $\forall \sigma \Rightarrow \sum a_\sigma e_\sigma = 0$ and ϕ is injective.

(iii) Suppose that $[f] = [g] \in H^2(G,L)$. That is $f/g \in \mathrm{im}\,\delta_1 \Rightarrow \exists h: G \to L^\times$ such that $f/g(\sigma,\tau) = \delta_1 h(\sigma,\tau) = \sigma h(\tau). h(\sigma\tau)^{-1}. h(\sigma)$.
$\Rightarrow f(\sigma,\tau) = \sigma h(\tau). h(\sigma\tau)^{-1}. h(\sigma). g(\sigma,\tau)$.

Define $V_f \xrightarrow{\phi} V_g$ by $e^f_\sigma \mapsto h(\sigma).e^g_\sigma$ where $\{e^f_\sigma\}_{\sigma\in G}$ is the basis of V_f and $\{e^g_\sigma\}_{\sigma \in G}$ that of V_g.

It is clear that ϕ has an inverse and so is bijective. It remains to see that it is a morphism of K-alg.

$$\begin{aligned}\phi(ae^f_\sigma \star be^f_\tau) &= \phi(a.\sigma(b).f(\sigma,\tau).e_{\sigma\tau}) \\ &= \phi(a.\sigma(b).\sigma h(\tau).h(\sigma\tau)^{-1}.h(\sigma).g(\sigma,\tau).e_{\sigma\tau}) \\ &= a.h(\sigma).\sigma(b.h(\tau)). g(\sigma,\tau).e_{\sigma\tau} \\ &= a.h(\sigma).e^g_\sigma \star b.h(\tau)e^g_\tau = \phi(a.e^g_\sigma) \star \phi(b.e^g_\tau).\end{aligned}$$

(iv) Recall that $Gal(\mathbb{C}/\mathbb{R}) = \{id, \sigma\}$ where σ is the complex conjugation.
We define the 2-cocycle $f: G \times G \to \mathbb{C}^\times$ as

$$f(id,id) = f(\sigma,id) = f(id,\sigma) = 1 \ \& \ f(\sigma,\sigma) = -1.$$

Then $V_f = \mathbb{C}x_{id} \oplus \mathbb{C}x_\sigma$ and the map $V_f \to \mathbb{H}$ given by
$cx_{id} + dx_\sigma \mapsto c + jd$ is an isomorphism.
Note that here $\mathbb{H}$ consists of numbers of the form $a+bi+cj+dk$ with $a,b,c,d \in \mathbb{R}$ & $i^2=j^2=k^2=-1$ and $k = ij = -ji$.